GOORMAGHTIGH'S EQUATION : SMALL PARAMETERS

MICHAEL A. BENNETT, BEN GARBUZ AND ADAM MARTENS

1. Introduction

The frequently studied polynomial-exponential equation of *Goormaghtigh*

$$\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1}, \quad m>n>2, \quad y>x\geq 2 \tag{1}$$

is conjectured to have precisely the solutions

$$(x,y,m,n)\in\{(2,5,5,3),(2,90,13,3)\} \tag{2}$$

in integers. Unconditionally, however, the number of such solutions is not known to be finite, even if one fixes one of the variables x, y or n (though recent work of the first author, Gherga and Kreso [4] establishes such a result for a given n under the additional assumption that $\gcd(m-1,n-1)>1$).

If one fixes any two of the variables x, y, m or n, however, then the number of solutions to (1) is, in fact, finite. This was proven for a given pair (x,y) by Kanold [13] and for fixed (m,n) by Davenport, Lewis and Schinzel [8]. Explicit versions of these results for small parameters (x,y) and (m,n) date back to work of Makowski and Schinzel [19], who proved the following pair of theorems.

Theorem 1 (Makowski-Schinzel)**.** *The only solution to equation (1) with*

$$2\leq x<y\leq 10 \tag{3}$$

is given by $(x,y,m,n)=(2,5,5,3)$.

Theorem 2 (Makowski-Schinzel)**.** *The only solution to equation (1) with* $m\leq 5$ *is given by* $(x,y,m,n)=(2,5,5,3)$.

The first of these theorems is of an entirely elementary nature, based upon congruential arguments, while the second applies the classical method of Runge [22]. Our goal in this paper is to introduce new techniques to improve the first of these results substantially, and to extend the second to take advantage of both improvements in computational power, and in the technical machinery underlying Runge's method. We prove the following.

Theorem 3. *If* (x,y,m,n) *is a a solution to (1) with*

$$2\leq x<y\leq 10^5,$$

then $(x,y,m,n)=(2,5,5,3)$ *or* $(2,90,13,3)$.

Date: January 24, 2019.

Theorem 4. *The only solutions to equation (1) with either*

$$m = n+1 \quad \text{and} \quad 3 \le n \le 17, \tag{4}$$

or

$$\gcd(m-1, n-1) > 1 \quad \text{and} \quad m \le 50 \tag{5}$$

or

$$(n, m) = (3, 6) \tag{6}$$

are given by $(x, y, m, n) = (2, 5, 5, 3)$ *or* $(2, 90, 13, 3)$.

Equation (1) has been the subject of much study. A good survey of results up to 2001 or so can be found in the paper of Shorey [23]. For more recent work, the reader may consult [11], [12], [16], [17], [18] and [28]. The current state of the art for applying techniques from Diophantine Approximation to (1) can be found in [4], and in the papers of Nesterenko and Shorey [20], and of Bugeaud and Shorey [7]. In the last of these, by way of example, one finds the following result.

Theorem 5 (Bugeaud-Shorey)**.** *Let* $\alpha > 1$. *Equation (1) with*

$$\gcd(m-1, n-1) > 4\alpha + 6 + \frac{1}{\alpha} \quad \text{and} \quad \frac{m-1}{n-1} \le \alpha$$

implies that $\max\{x, y, m, n\}$ *is bounded above by an effectively computable constant depending only on* α.

This theorem is a strong effective generalization of a classical result of Karanicoloff [14] which showed that $(x, y, m, n) = (2, 5, 5, 3)$ is the only solution to equation (1) with the property that $(m-1)/(n-1) = 2$. We apply the arguments leading to Theorem 5 to deduce

Theorem 6. *There are no solutions to equation (1) with* $(m-1)/(n-1) = 3$.

.

The outline of this paper is as follows. In Section 2, we prove Theorem 3. Theorem 4 is proved in Sections 3, 4 and 5. The first of these is devoted to equation (1) with (m, n) as in (4), the second to (m, n) satisfying (5), and the third to $(m, n) = (6, 3)$. In Section 6, we prove Theorem 6.

2. Fixed values of x and y : Theorem 3

In this section, we will prove Theorem 3. For fixed values of x and y, an explicit finiteness statement for solutions to equation (1) is provided by the following result of He and Togbé (Lemma 2.3 of [12]). This is a slight sharpening of earlier work of Bugeaud and Shorey [7] and is based upon bounds for linear forms in logarithms.

Lemma 2.1. *If* (x, y, m, n) *satisfy (1) with* $y > x \ge 2$, *then*

$$\frac{m-1}{1+\log m} < 1.391 \cdot 10^{11} (\log y)^2$$

For the remainder of this section, we will assume that $2 \le x < y \le 10^5$, whence it follows from Lemma 2.1 by routine calculation that if (x, y, m, n) satisfy equation (1), then

$$m < 10^{15} \tag{7}$$

Our main tool for handling the *a priori* roughly 10^{40} remaining tuples (x, y, m, n) is the following result, which is easily derived from equation (1) (see the proof of Lemma 2 of Bugeaud and Shorey [7]).

Lemma 2.2. *If (x, y, m, n) satisfy equation (1), then*

$$0 < m \log x - n \log y + \log \left(\frac{y-1}{x-1} \right) < 2 \cdot x^{-m}. \tag{8}$$

To reduce the number of cases under consideration to a manageable quantity, we will begin by proving the following.

Proposition 2.3. *Suppose that (x, y, m, n) is a a solution to (1) with*

$$2 \leq x < y \leq 10^5.$$

Then either $(x, y, m, n) = (2, 5, 5, 3)$ or $(2, 90, 13, 3)$, or we have that $m > 100$.

Proof. We consider the precisely 4753 pairs (n, m), with $3 \leq n < m \leq 100$. For each such pair, we have that

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \leq \frac{10^{5n} - 1}{10^5 - 1}$$

which provides an upper bound of the shape $x \leq x_0(n, m)$ (the largest of which corresponds to $x_0(99, 100) = 89021$). For each pair (m, n) and each x with $2 \leq x \leq x_0(m, n)$, we solve numerically the equation

$$n \log y - \log(y - 1) = m \log x - \log(x - 1)$$

and let y_0 denote the value of y we find with $x < y_0 \leq 10^5$. By the Mean Value Theorem, if there exists a solution to equation (1) corresponding to the given triple (m, n, x), with y an integer, we necessarily have, from inequality (13), that

$$|y - y_0| \leq \frac{2z(z-1)}{(nz - n - z) \cdot x^m}, \tag{9}$$

where z lies between y and y_0. It is easy to check that the right-hand side of inequality (9) is bounded above by 1, so that necessarily

$$\|y_0\| < \frac{2y_0(y_0+1)}{(n(y_0+1) - n - y_0 - 1) \cdot x^m}, \tag{10}$$

where $\|y_0\|$ denotes the distance to an integer of y_0. We verify that inequality (10) fails for all y_0 under consideration, except for $(x, m, n) = (2, 5, 3)$ or $(2, 13, 3)$, where we find that $y = 5$ and $y = 90$, respectively. This completes the proof of Proposition 2.3. The computation here took somewhat more than 100 hours in Maple, on a single core of a MacPro (2013 vintage), but is readily parallelized.

We note that, apart from these two examples, the closest we come to satisfying (10) is for $x = 5$, $n = 3$ and m even, corresponding to the family of "near"-solutions arising from the identity

$$\frac{\left(\frac{5^{m_0}-1}{2}\right)^3 - 1}{\frac{5^{m_0}-1}{2} - 1} = \frac{5^{2m_0} - 1}{5 - 1} + 1.$$

□

We next treat the cases with $\gcd(x, y) > 1$, appealing to Théorème 4 of Makowski and Schinzel [19].

Proposition 2.4 (Makowski-Schinzel). *If (x, y, m, n) satisfy (1) with $y > x \geq 2$ and $d = \gcd(x, y)$, then*

$$y \equiv x \pmod{d^n}. \tag{11}$$

Applying this result, for $2 \leq x < y \leq 10^5$ with $\gcd(x, y) = d > 1$, it follows that

$$10^5 \geq y > y - x \geq d^n,$$

and so

$$n \leq 16 \text{ if } d = 2 \text{ and } n \leq 10 \text{ if } d \geq 3.$$

If $x = d = 2$, we thus have

$$2^m - 1 = \frac{x^m - 1}{x - 1} \leq \frac{10^{80} - 1}{10^5 - 1}$$

and so $m \leq 249$. If $x = 4, d = 2$, then

$$\frac{4^m - 1}{3} = \frac{x^m - 1}{x - 1} \leq \frac{10^{80} - 1}{10^5 - 1},$$

and so $m \leq 125$. If $d = 2$ and $x \geq 6$,

$$\frac{6^m - 1}{5} \leq \frac{x^m - 1}{x - 1} \leq \frac{10^{80} - 1}{10^5 - 1},$$

and hence $m \leq 97$, contradicting Proposition 2.3. Similarly, if $d \geq 3$,

$$\frac{3^m - 1}{2} \leq \frac{x^m - 1}{x - 1} \leq \frac{10^{50} - 1}{10^5 - 1},$$

whence $m \leq 94$, again contradicting Proposition 2.3. We may thus suppose that either $x = d = 2$ and $101 \leq m \leq 249$, or that $x = 4, d = 2$ and $101 \leq m \leq 125$. A short computation as in the proof of Proposition 2.3 (only with x now fixed rather than y), with m in these ranges, $n \leq 16$ and $x \in \{2, 4\}$ leads, in each case, to a contradiction.

We may thus assume that $\gcd(x, y) = 1$. This already reduces the number of pairs (x, y) we need to consider to prove Theorem 3 from 4999850001 to 3039550754. We can, in fact, eliminate many more by appealing further to other elementary results of Makowski and Schinzel. The "best" of these for our purposes is Théorème 6 of Makowski and Schinzel [19].

Proposition 2.5 (Makowski-Schinzel). *Suppose that (x, y, m, n) satisfy (1) with $\gcd(x, y) = 1$. If a and b are coprime positive integers, denote by $ord_a(b)$ the smallest positive integer t with the property that $b^t \equiv 1 \pmod{a}$. Writing $\mu = ord_{xy-y}(x)$ and $\nu = ord_{xy-x}(y)$, we have that $\gcd\left(y^2, \frac{x^\mu - 1}{x-1}\right) = y$ and $\gcd\left(x^2, \frac{y^\nu - 1}{y-1}\right) = x$.*

A routine corollary of this result (combining Corollaires 1 and 2 of [19]) is the following.

Corollary 2.6. *Suppose that (x, y, m, n) satisfy (1). Then*

$$\gcd(y^2, x + 1) \mid y, \ \gcd(x^2, y + 1) \mid x, \ \gcd(y^2, x^2 + 1) \mid y \ \textit{ and } \ \gcd(x^2, y^2 + 1) \mid x.$$

Application of this corollary reduces the number of pairs (x, y) under consideration to 2099765696.

Other elementary results from [19] serve to eliminate more cases (we observe that the "smallest" pair not treated by the arguments of [19] is $(x, y) = (4, 11)$).

Since, for the values of (x,y) under consideration, these total rather less than 1% of the remaining cases, we will instead appeal to Lemma 2.2 which provides a computationally efficient way to search for small solutions (x,y) to (1), in conjunction at least with a version of a lemma of Baker and Davenport [3]. For the latter, we will use Lemma 5 of Dujella and Pethő [9]:

Lemma 2.7 (Dujella and Pethő)**.** *Suppose that M is a positive integer and that κ and μ are real numbers. Let p/q be a convergent in the infinite simple continued fraction expansion of κ satisfying $q > 6M$ and let*

$$\epsilon = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer.

If $\epsilon > 0$, and A and B are positive real numbers with $B > 1$, then there is no solution to the inequality

$$0 < m\kappa - n + \mu < A \cdot B^{-m} \tag{12}$$

in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

We will apply this result with

$$\kappa = \frac{\log x}{\log y}, \quad \mu = \frac{\log\left(\frac{y-1}{x-1}\right)}{\log y}, \quad A = \frac{2}{\log y}, \quad B = x \text{ and } M = 10^{15}.$$

For a given pair (x,y), our problem thus reduces to finding convergents p_k/q_k in the infinite simple continued fraction expansion to $\kappa = \frac{\log x}{\log y}$ which satisfy certain properties. Note that, since $\gcd(x,y) = 1$, such a κ is necessarily irrational. Suppose that, given (x,y) and a fixed real number $t > 20$, we can find a k such that we have

$$10^{20} \leq q_k < 10^t \quad \text{and} \quad \|\mu q_k\| > 10^{-4}. \tag{13}$$

Since any such convergent necessarily satisfies

$$\left|\kappa - \frac{p_k}{q_k}\right| < \frac{1}{q_k^2},$$

it follows that

$$\|\kappa q_k\| = |\kappa q_k - p_k| < \frac{1}{q_k},$$

whence, from (13) and $M = 10^{15}$, we thus have

$$\|\mu q\| - M \cdot \|\kappa q\| = \epsilon > 9 \cdot 10^{-5}.$$

From Lemma 2.7, (13) and inequality (7), it follows that

$$m < \frac{\log(Aq_k/\epsilon)}{\log B} = \frac{\log\left(\frac{2\cdot 10^{t+5}}{9\log y}\right)}{\log x}. \tag{14}$$

Since we may suppose that $y \geq 11$ and $x \geq 2$, if we take $t = 26$, the right-hand side here is smaller than 100, contradicting Proposition 2.3. Similarly, if we suppose that $x > 40$, then (14) contradicts Proposition 2.3 for all $t \leq 155$.

It remains then, to check to see if there exists a convergent p_k/q_k to $\kappa = \frac{\log x}{\log y}$ for which (13) holds, with $t = 26$ for $2 \leq x \leq 40$, and, say, $t = 40$, for $40 < x < 10^5$.

In case $\log x$ and $\log\left(\frac{y-1}{x-1}\right)$ are $\mathbb{Q}$-linearly dependent, this can fail to occur. Indeed, if there exists a positive integer s such that

$$\frac{y-1}{x-1} = x^s, \tag{15}$$

then $\mu = s\kappa$ and so

$$\|\mu q_k\| \leq |s\kappa q_k - s p_k| < \frac{s}{q_k} \leq \frac{16}{10^{20}} < 10^{-4}.$$

In this case, however, inequality (13) becomes

$$0 < (m+s)\log x - n \log y < 2 \cdot x^{-m}. \tag{16}$$

We may apply Corollary 2 of Laurent [15] to deduce a lower bound for this linear form with, in the notation of that paper, $m = 30$, $D = 1$ and

$$b' < \frac{2(m+s)}{\log y},$$

where this last inequality is a consequence of (13). From $s < \log y / \log x$, inequality (7) and $y \geq 11$, it follows from Corollary 2 of Laurent [15] that

$$\log|(m+s)\log x - n\log y| > -17.9 \cdot 34.74^2 \cdot \max\{\log x, 1\} \cdot \log y,$$

which, with (16), implies that

$$m < \frac{\log 2}{\log x} + 21603 \log y \text{ and so } m \leq 248714$$

if $x \geq 3$, and

$$m < 1 + 31167 \log y \text{ whence } m \leq 358824,$$

if $x = 2$. In any case, we have from (16) that

$$\left| \frac{\log x}{\log y} - \frac{n}{m+s} \right| < \frac{2}{(m+s)\log y \cdot x^m}$$

and hence, from $m \geq 101$ and $s \leq 16$, which together imply that

$$x^m \log y > 4(m+s),$$

we have that $\frac{n}{m+s} = \frac{p_k}{q_k}$ for some convergent p_k/q_k to $\frac{\log x}{\log y}$. Further, since

$$\left| \frac{\log x}{\log y} - \frac{p_k}{q_k} \right| > \frac{1}{(a_{k+1}+2)q_k^2},$$

where a_{k+1} is the corresponding $(k+1)$st partial quotient, it follows that

$$a_{k+1} > \frac{\log(y) \cdot x^m}{2(m+s)} - 2 \geq \frac{\log(11) \cdot 2^{101}}{234} - 2 > 10^{28},$$

while $q_k \leq m + s \leq 358840$. A routine computation of the 409 continued fractions involved shows that this does not occur.

We may thus suppose that identity (15) is not satisfied for any positive integer s. For the remaining roughly 2×10^9 pairs (x, y), we used code written in Maple to calculate the simple continued fraction of κ and verify that there exists a convergent p_k/q_k to $\kappa = \frac{\log x}{\log y}$ for which (13) holds, with $t = 26$ for $2 \leq x \leq 40$, and $t = 40$, for $40 < x < 10^5$. This is easy to do in parallel, though not an especially short computation (taking approximately 2000 hours of processor time). Full details (and the relevant p_k/q_k) are available from the authors upon request. The storage of these convergents takes roughly 100 gigabytes of memory. This completes the proof of Theorem 3.

3. The case $\gcd(m-1, n-1) > 1$

We next turn our attention to equation (1) where the exponents m and n have the property that

$$\gcd(m-1, n-1) = d > 1. \tag{17}$$

As it transpires, this condition allows one to apply a wide variety of effective methods from Diophantine Approximation to the problem, including lower bounds for linear forms in logarithms, Runge's method and the hypergeometric method of Thue and Siegel. Combining results from [4] and [20], we have

Theorem 7 (Nesterenko-Shorey, B-Gherga-Kreso)**.** *If there is a solution in integers x, y, n and m to equation (1), satisfying (17), then*

$$x < (3d)^{4n/d} \tag{18}$$

and

$$x < \max\left\{9, 1 + \frac{1}{2}(d+1)d^{r-2}\prod_{p|d} p^{\nu_p(r!)}\right\}, \tag{19}$$

where $r = (m-1)/d$ and $\nu_p(t)$ is the largest power of p dividing a given positive integer t.

Theorem 8 (B-Gherga-Kreso)**.** *If there is a solution in integers x, y and m to equation (1), with $n \in \{3, 4, 5\}$ and satisfying (17), then*

$$(x, y, m, n) = (2, 5, 5, 3) \text{ and } (2, 90, 13, 3).$$

From these results, to prove Theorem 4, we may thus suppose that $6 \leq n < m \leq 50$ and that $\gcd(m-1, n-1) > 1$.

an algorithm described in the paper of S. Tengely [25].

This algorithm relies crucially on a condition known as *Runge's Condition* [22], which is as follows

Proposition 3.1 (Runge's Condition)**.** *Let $P(X, Y) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}X^iY^j$ be an irreducible polynomial over $\mathbb{Q}$ with coefficients in $\mathbb{Z}$. P is said to satisfy Runge's Condition unless all of the following holds:*

1. *$a_{iN} = a_{Mj} = 0$ for all non-zero i, j*
2. *for every monomial $a_{ij}X^iY^j$ of P, $Ni + Mj \geq MN$*
3. *the sum of all monomial of P for which $Ni + Mj = NM$ is a constant multiple of a power of an irreducible polynomial in $\mathbb{Z}[X, Y]$.*
4. *there is only one system of conjugate Puiseux expansions at $x = \infty$ for the algebraic function $y = y(x)$ defined by $P(x, y) = 0$.*

If P satisfies Runge's condition, then $P(x, y) = 0$ has finitely many rational solutions $(x, y) \in \mathbb{Q}^2$ (and hence finitely many integer solutions).

From Goormaghtigh's equation, we get

$$P(X,Y)=\sum_{i=1}^{m-1}X^i-\sum_{j=1}^{n-1}Y^j=0 \tag{20}$$

which clearly satisfies Runge's condition, since the sum of all monomials satisfying $(n-1)i+(m-1)j=(m-1)(n-1)$ is simply 0. Therefore, for fixed (m,n), we are guaranteed finitely many rational (and hence integer) solutions to Goormaghtigh's equation.

The algorithm also requires a lemma, found in [25], the proof of which can be found in [26].

Lemma 3.2. *Let $F(X)$, $G(Y)$ be two polynomials with degrees M,N respectively (in our case, $M=m-1, N=n-1$). Suppose that $\gcd(M,N)>1$. Let d be a divisor of $\gcd(M,N)$. Then there exist Laurent expansions*

$$u(X)=\sum_{i=-\frac{n}{d}}^{\infty}f_iX^i$$

$$v(Y)=\sum_{i=-\frac{m}{d}}^{\infty}g_iY^i$$

of the algebraic functions U and V defined by $U^d=F(X)$ and $V^d=G(Y)$, such that $d^{2(N/d+i)-1}f_i\in\mathbb{Z}$ and $d^{2(M/d+i-1)}g_i\in\mathbb{Z}$ for all i, and $f_{-\frac{N}{d}}=g_{-\frac{M}{d}}=1$. Furthermore, $|f_i|\le(H(F)+1)^{\frac{N}{d}+i+1}$ and $|g_i|\le(H(G)+1)^{\frac{M}{d}+i+1}$ where $H(\cdot)$ is the height of a polynomial.

We describe the algorithm, for the case where $\gcd(m-1,n-1)$ is odd; the even case requires that one be slightly more careful, and is covered in the original paper [25].

Let U and V be as in Lemma 3.2, with $F(Y)=\frac{X^m-1}{X-1}, G(Y)=\frac{Y^n-1}{Y-1}$. Let p be the smallest prime divisor of $\gcd(m-1,n-1)$ (in this case, $p\ge3$). Let D be the least common multiple of the denominators of the coefficients of the non-constant terms of U and V, as well as the denominator of the difference of their constant terms, and let $t>0$. $F(x)-(U(x)+t)^p$ and $F(x)-(U(x)-t)^p$ have opposite signs, with their degrees being even. Thus, for sufficiently large $|x|$, we have $(U(x)-t)^p<F(x)<(U(x)+t)^p$. A similar situation applies to G,V and y. Let $[x_t^-,x_t^+]$ be the interval where x_t^- is the smallest (real) root of $F(x)-(U(x)-t)^p$, and x_t^+ is the largest (real) root of $F(x)-(U(x)+t)^p$. Let $[y_t^-,y_t^+]$ be the analogous interval for y.

If (x,y) is an integer solution to Goormaghtigh's equation with neither $x\in[x_t^-,x_t^+]$ nor $y\in[y_t^-,y_t^+]$, we can say that

$$(U(x)-t)^p-(V(y)+t)^p<F(x)-G(y)<(U(x)+t)^p-(V(y)-t)^p.$$

Following a series of manipulations, it turns out that

$$-2t < u(x) - v(y) < 2t.$$

From Lemma 3.2 we know that the denominator of $u(x)-v(y)$ divides $p^{\frac{2m}{p}-1}$, so that also D divides $p^{\frac{2m}{p}-1}$. Therefore x is a solution to $\mathrm{Res}_Y(F(X)-G(Y), U(X)-V(Y)-T) = 0$ for some rational T, $|T| < 2t$, and the denominator of T dividing D. Our algorithm is thus as follows:

First, pick a t; Tengely describes a way of choosing a good value in his paper. Then, search for integer solutions to Goormaghtigh's equation for $x \in [x_t^-, x_t^+]$ and for $y \in [y_t^-, y_t^+]$. After this, find all integer solutions x of the equation $\mathrm{Res}_Y(F(X)-G(Y), U(X)-V(Y)-T) = 0$ for all rational numbers T with $|T| < 2t$ and the denominator of T dividing D. For all solutions found from the resultant equation, find the corresponding y.

We implemented the algorithm given by Tengely, written in Maple, to verify that, indeed, no integer solutions to Goormaghtigh's equation exist for $2 < n < m \leq 50$ given that $\gcd(m-1, n-1) > 1$.

4. The case $m = n + 1$

We now turn to the case where $m = n + 1$. From Theorem 2, we may suppose that $n \geq 5$. Again Runge's method turns out to be applicable (to an auxiliary equation) and, in fact, works particularly well, since here the Puisseux expansions are actually Laurent expansions with positive coefficients. We start, as in Davenport, Lewis, and Schinzel [8] (this argument has its genesis in the paper of Makowski and Schinzel [19]), with the fact that we can re-write Goormaghtigh's equation for $m = n + 1$ as

$$x^n = (y-x)\sum_{k=1}^{n-1} \frac{y^k - x^k}{y - x}. \tag{21}$$

It follows that there exist positive integers a and b such that $y - x = a^n$, $\sum\limits_{k=1}^{n-1} \frac{y^k - x^k}{y-x} = b^n$, and $ab = x$. Substituting these back into (21), we are led to the equation

$$F(a,b) = b^n - \sum_{j=0}^{n-2} \left(\sum_{i=0}^{n-j-2} \binom{i+j+1}{j} a^{j+ni} \right) b^j = 0. \tag{22}$$

From this, we may write b as a Laurent series expansion in a,

$$b = a^{n-2} + \frac{p_{n-3}}{q_{n-3}} a^{n-3} + \cdots + \frac{p_1}{q_1} a + \frac{p_0}{q_0} + \frac{p_{-1}}{q_{-1}} a^{-1} + \cdots ,$$

where the p_i and q_i are positive integers, with $\gcd(p_i, q_i) = 1$, for each i. We may check that, at least for the values of n under consideration, we always have that $q_i \mid q_0$, for $i = 1, 2, \ldots, n-3$ and hence

$$P_n(a) = q_0 a^{n-2} + \frac{q_0 p_{n-3}}{q_{n-3}} a^{n-3} + \cdots + \frac{q_0 p_1}{q_1} a + p_0 \tag{23}$$

is a polynomial in a with positive integer coefficients. If we have a solution to (22) in positive integers a and b, then necessarily $q_0 b = P_n(a) + k$ for some positive integer k, where we have, additionally, that

$$P_n(a) \equiv -k \pmod{q_0}. \tag{24}$$

Let us now define, for each positive integer k,

$$G_k(a) = (P_n(a)+k)^n - \sum_{j=0}^{n-2} \left(\sum_{i=0}^{n-j-2} \binom{i+j+1}{j} a^{j+ni} \right) q_0^{n-j} (P_n(a)+k)^j.$$

If there exists a solution to (22) in positive integers a and b, then, necessarily, there exists a solution to $G_k(a) = 0$ in positive integers a and k, satisfying (24). Defining $a_{k,n}$ to be the largest real number for which $G_k(a_{k,n}) = 0$, it follows, via calculus, that $a_{k,n}$ is decreasing in k and that $G_k(a) < 0$ for $a > a_{k,n}$.

To illustrate how we may turn these observations into an efficient algorithm for solving equation (22), let us consider the case $n = 6$. Here, we have

$$P_6(a) = 31104a^4 + 25920a^3 + 19440a^2 + 13440a + 8645,$$

$q_0 = 31104$ and $a_{1,6} = 61.52146\ldots$. It follows immediately that we have that $a \leq 61$. For these values of a, a short computation reveals that, from (24), we have $k \geq 379$ (corresponding to $a = 46$). Since $a_{379,6} = 3.418385\ldots$, we thus have $1 \leq a \leq 3$. A short check that $F(a,b)$ has no integral roots for these values of a completes this case.

For larger values of n, we actually argue somewhat differently. Let us illustrate this in case $n = 16$. We find that

$$a_{1,16} \sim 2.75 \times 10^9 \quad \text{and} \quad q_0 = 147573952589676412928.$$

Rather than looping through this (large) collection of values of a, we instead solve the congruence (24), to find that the only solutions with $k < k_0 = 4 \times 10^7$ are with

$$k \in \{2445, 6541, 10637, 14221, 14733, 18829, \ldots, 39999885\},$$

a set with precisely 13023 elements. In each case, $a_{k,16}$ is not an integer and we find that, in each case, the smallest positive solution a to the congruence (24) for any of these values k exceeds $a_{1,16}$. Since $a_{4\times 10^7,16} < 435359$, we may thus conclude that $1 \leq a \leq 435358$. For each of these values, we check to see whether or not the polynomial $F(a,b)$ has an integer root. The total time for this computation was somewhat less than eight hours on a single core in Maple. The only case when the resulting polynomial (in b) failed to be irreducible was with $a = 1$, where we find the root $b = -1$.

A program was written in Maple to implement the above algorithm, and with it we were able to show that there are no solutions to the $m = n + 1$ case for $3 \leq n \leq 17$; the only roots encountered for $F(a,b)$ corresponded to $(a,b) = (1,-1)$. We tabulate our data as follows.

n	$[a_{1,n}]$	k_0	#	$[a_{k_0,n}]$	time
5	4	20	4	1	$< 1s$
6	61	20	0	13	$< 1s$
7	42	20	3	9	$< 1s$
8	627	20	0	140	$< 1s$
9	1909	1000	111	60	$< 1s$
10	70325	4000	5	1112	$3s$
11	12954	4000	364	205	$2s$
12	9205553	6×10^5	1042	11884	$354s$
13	332194	5×10^4	3847	1485	$28s$
14	153170043	3×10^6	4463	88433	$1382s$
15	801682738	4×10^6	266666	400841	$10044s$
16	2753445124	4×10^7	13023	435358	$27400s$
17	373406096	3×10^6	176471	215586	$8490s$

For larger values of n, the computation rapidly becomes rather unwieldy.

5. The case $(n, m) = (3, 6)$

In case $(n, m) = (3, 6)$, equation (1) becomes

$$y^2 + y + 1 = x^5 + x^4 + x^3 + x^2 + x + 1$$

whereby $Y = 16(2y + 1)$ and $X = 4x$ satisfies

$$Y^2 = X^5 + 4X^4 + 16X^3 + 64X^2 + 256X + 256. \tag{25}$$

We would like to show that the only rational solutions to this equation are with $X = 0$. Appealing to Magma [5], we use the following commands

```
_<x> := PolynomialRing(Rationals());
> C := HyperellipticCurve(x^5+4*x^4+16*x^3+64*x^2+256*x+256);
> ptsC := Points(C : Bound:=1000); ptsC;
{@ (1 : 0 : 0), (0 : -16 : 1), (0 : 16 : 1) @}
> J := Jacobian(C);
> RankBound(J);
1
> TorsionSubgroup(J);
Abelian Group of order 1
> PJ := J! [ ptsC[2], ptsC[1]];
> Order(PJ);
0
```

to deduce that the Jacobian of the curve corresponding to (25) has Mordell-Weil group with rank 1 and trivial torsion, and that the point we are calling PJ has infinite order in this group. The commands

```
> Height(PJ);
0.0594215465492475716871323583279
> LogarithmicBound := Height(PJ)+HeightConstant(J);
> AbsoluteBound := Ceiling(Exp(LogarithmicBound));
> PtsUpToAbsBound := Points(J : Bound:=AbsoluteBound );
> ReducedBasis( [ pt : pt in PtsUpToAbsBound ]);
[ (x, -16, 1) ]
```

```
[0.0594215465492475716871323583279]
```

then show that PJ is in fact a generator of the Mordell-Weil group. Finally, applying a Chabauty argument with $p = 7$

```
> BadPrimes(C);
[ 2, 3, 23 ]
> Chabauty(PJ,7);
{@ <0, 1, 4, 1> @}
```

leads to the desired conclusion. This complete the proof of Theorem 4.

6. The case $(m-1) = 3(n-1)$

Finally, we will focus our attention on the situation when the ratio $\frac{m-1}{n-1}$ is a small fixed positive integer. In 1963, Karanicoloff [14] showed that the only solution to (1) with $\frac{m-1}{n-1} = 2$ is given by $(x, y, m, n) = (2, 5, 5, 3)$. We will treat the case $k = 3$ with a rather different argument.

To start, we will appeal to a result of Bugeaud and Shorey [7], whose proof is based upon lower bounds for linear forms in logarithms.

Lemma 6.1. *Let (x, y, m, n) be a solution of (1). Then we have*

$$\gcd(m-1, n-1) \leq 33.4\, m^{1/2}.$$

From this, if $\frac{m-1}{n-1} = 3$, it follows that $n \leq 3348$. Further, from Theorems 3 and 4, we may suppose that $n \geq 18$ and that $x \geq 47$. In this case, from equation (1) with $m = 3n-2$, we may write y as a Laurent expansion in terms of x :

$$y = x^3 + \frac{x^2}{n-1} + \left(\frac{n}{2(n-1)^2}\right) x + \frac{2n^2-n}{6(n-1)^3} + \frac{6n^3-7n^2+2n}{24(n-1)^4 x} + \sum_{k=2}^{\infty} E_k(x,n) \tag{26}$$

where each $E_k(x,n)$ is positive and bounded above by

$$F_k(x,n) = \frac{n^{k+2}}{(k+3)(n-1)^{k+3} x^k}.$$

Since $n \geq 18$, we have

$$\frac{F_k(x,n)}{F_{k+1}(x,n)} = \frac{(k+4)(n-1)x}{(k+3)n} > \frac{17x}{18},$$

whence, from $x \geq 47$,

$$\sum_{k=2}^{\infty} E_k(x,n) < 1.1\, E_2(k,n) < 1.1\, \frac{n^4}{5(n-1)^5 x^2} < \frac{0.28}{(n-1)\, x^2}.$$

We may thus write

$$y = x^3 + \frac{x^2}{n-1} + \left(\frac{n}{2(n-1)^2}\right) x + \frac{2n^2-n}{6(n-1)^3} + E(x,n), \tag{27}$$

where

$$0 < E(x,n) < \frac{1}{3(n-1)x}.$$

It follows that

$$\left\| \frac{x^2}{n-1} + \left(\frac{n}{2(n-1)^2}\right) x + \frac{2n^2-n}{6(n-1)^3} \right\| < \frac{1}{3(n-1)x}. \tag{28}$$

Since inequality (19) implies that

$$x \leq 3n(n-1),$$

it remains to check to see whether or not inequality (28) is satisfied for each n with $18 \leq n \leq 3348$ and each x with $47 \leq x \leq 3n(n-1)$.

A Maple calculation (of roughly 80 hours on a single core) verifies, for the values of n and x under consideration, that (28) holds precisely when $n \equiv 1 \pmod 6$ and $x = x_0$ for

$$x_0 = \frac{1}{3}\left((2j-1)n^2 - (4j+3)n + 2j + 1\right).$$

Here, $j \in \{1,2,3\}$ and $n \equiv 6j - 11 \pmod{18}$. Defining

$$f(x,n) = 6(n-1)^2x^2 + 3n(n-1)x + 2n^2 - n,$$

it follows, after a little work, that $\frac{f(x_0,n)}{6(n-1)^3}$ is equal to

$$\frac{(2j-1)^2n^3}{9} - \frac{(12j^2+8j-7)n^2}{9} + \frac{(24j^2+38j+25)n}{18} - \frac{8j^2+8j-1}{18} + \frac{1}{6(n-1)^3},$$

whence

$$\left\| \frac{x_0^2}{n-1} + \left(\frac{n}{2(n-1)^2}\right)x_0 + \frac{2n^2-n}{6(n-1)^3} \right\| = \frac{1}{6(n-1)^3} < \frac{1}{3(n-1)x_0}.$$

Notice that if $n \equiv 1 \pmod 6$ and x_0 corresponds to a solution to (1) with $m = 3n-2$, then we necessarily have

$$E(x_0,n) = \frac{1}{6(n-1)^3},$$

so that

$$\frac{6n^3-7n^2+2n}{24(n-1)^4x_0} + \sum_{k=2}^{\infty} E_k(x_0,n) = \frac{1}{6(n-1)^3},$$

whence, arguing as previously,

$$\frac{6n^3-7n^2+2n}{24(n-1)^4x_0} < \frac{1}{6(n-1)^3} < \frac{6n^3-7n^2+2n}{24(n-1)^4x_0} + \frac{0.28}{(n-1)\,x_0^2}. \tag{29}$$

On the other hand, since $n \geq 18$, we may readily show that

$$\frac{6n^3-7n^2+2n}{24(n-1)^4x_0} > \frac{1}{6(n-1)^3}$$

for $j \in \{1,2\}$, and that

$$\frac{6n^3-7n^2+2n}{24(n-1)^4x_0} < \frac{1}{6(n-1)^3}$$

for $j = 3$ and $n \geq 43$, in each case contradicting (29). Finally, if $n = 25$ and $x_0 = 919$, we simply check that equation (1) fails to be satisfied. This completes the proof of Theorem 6.

References

[1] R. Balasubramanian and T. N. Shorey, On the equation $a(x^m-1)/(x-1) = b(y^n-1)/(y-1)$, *Math. Scand.* 46 (1980), 177–182.

[2] A. Baker, Bounds for the solutions of the hyperelliptic equation, *Math. Proc. Camb. Phil. Soc.*. 65 (1969), 439–444.

[3] A. Baker and H. Davenport, The equations $3x^2-2=y^2$ and $8x^2-7=z^2$, *Quart. J. Math. Oxford Ser. (2)* 20 (1969), 129–137.

[4] M. A. Bennett, A. Gherga and D. Kreso, An old and new approach to Goormaghtigh's equation, submitted for publication.

[5] W. Bosma, J. Cannon and C. Playoust, The Magma Algebra System I: The User Language, *J. Symb. Comp.* **24** (1997), 235–265. (See also `http://magma.maths.usyd.edu.au/magma/`)

[6] Y. Bugeaud and K. Győry, Bounds for the solutions of Thue-Mahler equations and norm form equations, *Acta Arithmetica* 74.3 (1996), 273–292.

[7] Y. Bugeaud and T.N. Shorey, On the diophantine equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, *Pacific J. Math.* 207 (2002), 61–75.

[8] H. Davenport, D. J. Lewis and A. Schinzel, Equations of the form $f(x) = g(y)$, *Quart. J. Math. Oxford set (2)* 12 (1961), 304–312.

[9] A. Dujella and A. Pethő, Generalization of a theorem of Baker and Davenport, *??*

[10] R. Goormaghtigh, *L'Intermédiaire des Mathématiciens* 24 (1917), 88.

[11] B. He, A remark on the Diophantine equation $(x^3-1)/(x-1) = (y^n-1)/(y-1)$, *Glasnik Mat.* 44 (2009), 1–6.

[12] B. He and A. Togbé, On the number of solutions of Goormaghtigh equation for given x and y, *Indag. Mathem.* 19 (2008), 65–72.

[13] H.-J. Kanold, ¨ber einen Satz von L. E. Dickson. II, *Math. Ann.* 132 (1956), 246–255.

[14] C. Karanicoloff, Sur une équation diophantienne considérée par Goormaghtigh, *Ann. Polonici Math.* XIV (1963), 69–76.

[15] M. Laurent, Linear forms in two logarithms and interpolation determinants II, *Acta Arith.* 133 (2008), 325–348.

[16] M. Le, On the Diophantine equation $(x^3-1)/(x-1) = (y^n-1)/(y-1)$, *Trans. Amer. Math. Soc.* 351 (1999), 1063–1074.

[17] M. Le, Exceptional solutions to the exponential Diophantine equation $(x^3-1)/(x-1) = (y^n-1)/(y-1)$, *J. Reine Angew. Math.* 543 (2002), 187–192.

[18] M. Le, On Goormaghtigh's equation $(x^3-1)/(x-1) = (y^n-1)/(y-1)$, *Acta Math. Sinica (Chin. Ser.)* 45 (2002), 505–508.

[19] A. Makowski and A. Schinzel Sur l'équation indéterminée de R. Goormaghtigh, *Mathesis* 68 (1959), 128-142.

[20] Y. V. Nesterenko and T. N. Shorey, On an equation of Goormaghtigh, *Acta Arith.* 83 (1998), 381–389.

[21] R. Ratat, *L'Intermédiaire des Mathématiciens* 23 (1916), 150.

[22] C. Runge, U ber ganzzahlige L osungen von Gleichungen zwischen zwei Ver ander-lichen, J. Reine Angew. Math. 100 (1887), 425–435.

[23] T. N. Shorey, An equation of Goormaghtigh and Diophantine approximations, Current Trends in Number Theory, edited by S.D.Adhikari, S.A.Katre and B.Ramakrishnan, Hindustan Book Agency, New Delhi (2002), 185–197.

[24] T. N. Shorey and R. Tijdeman, New applications of diophantine approximation to diophantine equations, *Math. Scand.* 39 (1976), 5–18.

[25] S. Tengely, Effective Methods for Diophantine Equations, *Not sure where this is published* 13 (2005), 11–22.

[26] P.G. Walsh, A quantitative version of Runge's theorem on Diophantine equations, *Acta Arith.,* 62(2) (1992), 157–172.

[27] P. Yuan, On the Diophantine equation $ax^2 + by^2 = ck^n$, *Indag. Mathem.* 16 (2) (2005), 301–320.

[28] P. Yuan, On the diophantine equation $\frac{x^3-1}{x-1} = \frac{y^n-1}{y-1}$, *J. Number Theory* 112 (2005), 20–25.